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APPLICATION OF FREE VIBRATION ANALYSIS OF MEMBRANES USING THE NON-DIMENSIONAL DYNAMIC INFLUENCE FUNCTION

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In this paper, the multi-domain method of subdividing the membranes of interest into several domains is presented for applications of free vibration analysis of arbitrarily shaped membranes. The method is especially effective for concavely shaped membranes with high concavity and multi-connected membranes with a hole, compared with the single-domain method. The frequency equations of these membranes are given by the determinants of the system matrices obtained when the compatibility conditions are considered on the common boundaries along which the sub-domains adjoin each other. Although many boundary nodes are distributed along the fixed boundaries on which transverse displacements are prescribed to be zero, the number of boundary nodes has no effect on the order of the system matrix. The order is determined to be equal to the value corresponding to twice the number of interior nodes distributed on the common boundaries. Thus, eigenvalues, the roots of the frequency equation, may be obtained from the determinant of the system matrix of a small order. The case studies presented in the paper reveal that the eigenvalues calculated by this method are in good agreement with those obtained by the FEM program, ANSYS.

1. INTRODUCTION

In a previous paper [1], the authors proposed a practical method using the non-dimensional dynamic influence function for free vibration analysis of arbitrarily shaped membranes. The method is remarkable in that it is very simple because no interpolation function is used between nodes, unlike the finite element method [2] and the boundary element method [3, 4]. Since the method uses no interpolation function, it employs the collocation technique or point-matching method [5, 6] when a given boundary condition is considered at the edges of the membrane of interest. However, it is revealed in the present work that the method does not give good solutions for concavely shaped membranes with high concavity, and that the method may not be essentially extended immediately to multi-connected membranes with holes.

This paper introduces the multi-domain method of subdividing the total domain of a membrane into sub-domains. In this method, system matrix equations for each of the sub-domains are constructed according to the single-domain method that was developed in the previous research [1]. The relationship between the equations is given by considering the compatibility conditions on common boundaries between the sub-domains. Finally, a frequency equation governing the total domain is obtained from the determinant of a new system matrix of a small order. Cortinez and Laura [7] studied the free vibration of

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a non-homogeneous rectangular membrane by applying the compatibility conditions to the boundary between two rectangular sub-domains, in the same manner as in the present work. However, applications of his study are limited to membranes with exact solutions, such as rectangular or circular membranes. Irie *et al.* [8] calculated the natural frequencies of concavely shaped polygonal membranes by using series-type solutions, but this method has the limitation that it is applicable only to polygonal membranes. Although many engineering applications have dealt with the vibration problems of membranes, a survey of the literature performed by the authors reveals that no paper associated with the topic of this paper has been published. Moreover, no papers have been presented for arbitrarily shaped membranes with holes.

In general, the Bessel functions of the first and second kinds of order n, J_n and Y_n, are required to solve free vibrations of membranes with a hole. However, the proposed method enables us to calculate the eigenvalue of the membranes with only the Bessel function of the first kind and order zero, J_0. Although the method uses a very small number of boundary nodes, compared with that used in the FEM, it gives accurate eigenvalues owing to its simplicity in that no interpolation function is used between nodes.

2. THE SINGLE-DOMAIN METHOD REVIEWED

2.1. NON-DIMENSIONAL DYNAMIC INFLUENCE FUNCTION

The *non-dimensional dynamic influence function* (*NDIF*) primarily satisfies the governing equation of the eigenfield of interest, and physically describes the displacement response of a point in an infinite domain due to a unit displacement excited at another point [1]. In the case of an infinite membrane (see Figure 1), the *NDIF* between the excitation point P and the response point P_k is given by

$$NDIF = J_0(\Lambda|\mathbf{r} - \mathbf{r}_k|), \tag{1}$$

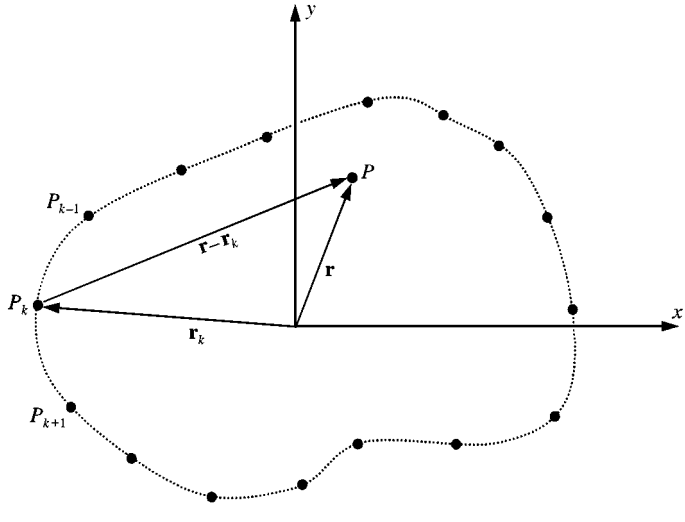

Figure 1. Infinite membrane with harmonic excitation points that are distributed along the fictitious contour with the same shape as the finite-sized membrane of interest.

which satisfies the Helmholtz equation

$$\nabla^2 W + \Lambda^2 W = 0, \tag{2}$$

where $W(\mathbf{r})$ is the transverse displacement of a finite-sized membrane, $\Lambda = \omega/\sqrt{T/\rho}$ denotes the wavenumber expressed in terms of the angular frequency ω, the uniform tension per unit length T, and the mass per unit area ρ. Detailed illustrations on the *NDIF* have been given in the previous paper [1].

2.2. SYSTEM EQUATION OF A SINGLE EIGENFIELD

For free vibration analysis of an arbitrarily shaped membrane, the boundary of which is illustrated by the dotted line in Figure 1, N nodes are first distributed along the boundary depicted in an infinite membrane. Assuming that harmonic displacements of amplitudes A_1, $A_2, \ldots, A_N$ are, respectively, generated at points $P_1, P_2, \ldots, P_N$, the total displacement response at the point P may be obtained by the sum of responses that have resulted from each boundary point, i.e.,

$$W(\mathbf{r}) = \sum_{k=1}^{N} A_k \mathrm{J}_0(\Lambda|\mathbf{r} - \mathbf{r}_k|), \tag{3}$$

which is employed as an approximate solution for the eigenfield of the finite-sized membrane. Note that the approximate solution also satisfies the Helmholtz equation.

Applying a boundary condition, which is prescribed continuously along the boundary, to the approximate solution is based on the collocation technique. Thus, the continuous boundary condition is discretized at the boundary points, i.e., as a discrete boundary condition, displacements at points $P_1, P_2, \ldots, P_N$ are given by $U_1, U_2, \ldots, U_N$ respectively. Then, the newly constructed discrete boundary condition

$$W(\mathbf{r}_i) = U_i, \quad i = 1, 2, \ldots, N \tag{4}$$

is applied to the approximate solution, equation (3):

$$W(\mathbf{r}_i) = \sum_{k=1}^{N} A_k \mathrm{J}_0(\Lambda|\mathbf{r}_i - \mathbf{r}_k|) = U_i, \quad i = 1, 2, \ldots, N, \tag{5}$$

which may be written in a simple matrix form as

$$\mathbf{SM}(\Lambda)\mathbf{A} = \mathbf{U}, \tag{6}$$

where the $N \times N$ symmetric system matrix $\mathbf{SM}(\Lambda)$ is given by $SM_{ik} = \mathrm{J}_0(\Lambda|\mathbf{r}_i - \mathbf{r}_k|)$, the participation vector $\mathbf{A}$ represents the participation strength of the *NDIF*'s defined at each boundary point, and the displacement vector $\mathbf{U}$ represents the discrete boundary condition. Equation (6) is termed the system matrix equation obtained by using the single-domain method for which the membrane of interest is not subdivided. In the case of $\mathbf{U} = \mathbf{0}$ (fixed boundary condition), eigenvalues can be found from the fact that $\det[\mathbf{SM}(\Lambda)] = 0$.

3. MULTI-DOMAIN METHOD

When the single-domain method is applied to concave or multi-connected membranes of complex shapes, too many boundary points may be required to well depict the shapes. In this case, the order of the system matrix becomes very large, so that it may be difficult to calculate the determinant of the system matrix owing to numerical errors. Thus, the multi-domain method of dividing the membrane of interest into several domains is presented to reduce the order of the system matrix.

3.1. TWO-DOMAIN METHOD: MEMBRANE WITH HIGH CONCAVITY

As shown in Figure 2, a concave membrane, illustrated as the solid line, is divided into the two domains, D_{I} and D_{II}, which are surrounded by boundaries $\Gamma_1 + \Gamma_a$ and $\Gamma_2 + \tilde{\Gamma}_a$ respectively. Note that $\tilde{\Gamma}_a$ is at the same position as Γ_a. When the single-domain method is applied to each domain, the two system matrix equations

$$\mathbf{SM}_{\mathrm{I}}\mathbf{A}_{\mathrm{I}} = \mathbf{U}_{\mathrm{I}}, \qquad \mathbf{SM}_{\mathrm{II}}\mathbf{A}_{\mathrm{II}} = \mathbf{U}_{\mathrm{II}} \tag{7, 8}$$

are obtained. Equations (7, 8) may be, respectively, written in another form:

$$\begin{bmatrix} \mathbf{SM}_{11} & \mathbf{SM}_{1a} \\ \mathbf{SM}_{a1} & \mathbf{SM}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_1 \\ \mathbf{A}_a \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_1 \\ \mathbf{U}_a \end{Bmatrix}, \qquad \begin{bmatrix} \mathbf{SM}_{22} & \mathbf{SM}_{2a} \\ \mathbf{SM}_{a2} & \mathbf{SM}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_2 \\ \tilde{\mathbf{A}}_a \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_2 \\ \tilde{\mathbf{U}}_a \end{Bmatrix} \tag{9, 10}$$

or

$$\mathbf{SM}_{11}\mathbf{A}_1 + \mathbf{SM}_{1a}\mathbf{A}_a = \mathbf{U}_1, \qquad \mathbf{SM}_{a1}\mathbf{A}_1 + \mathbf{SM}_{aa}\mathbf{A}_a = \mathbf{U}_a, \tag{11, 12}$$

$$\mathbf{SM}_{22}\mathbf{A}_2 + \mathbf{SM}_{2a}\mathbf{A}_a = \mathbf{U}_2, \qquad \mathbf{SM}_{a2}\mathbf{A}_2 + \mathbf{SM}_{aa}\tilde{\mathbf{A}}_a = \tilde{\mathbf{U}}_a, \tag{13, 14}$$

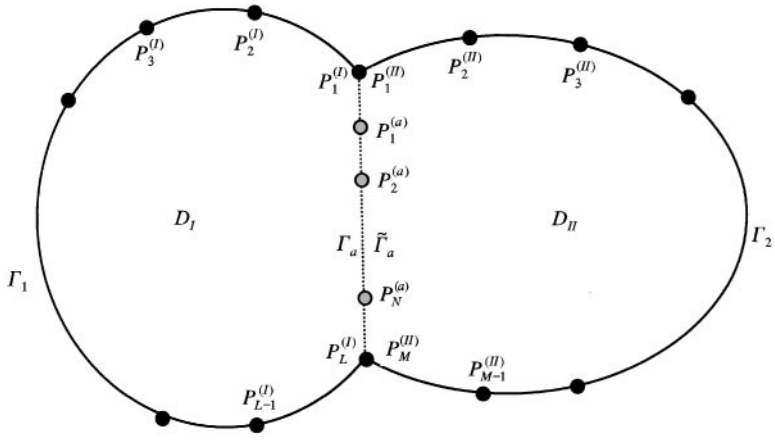

Figure 2. Concavely shaped membrane subdivided into two domains D_{I} and D_{II}, surrounded, respectively, by the fixed and common boundaries, $\Gamma_1 + \Gamma_a$ and $\Gamma_2 + \tilde{\Gamma}_a$.

where the vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ are given from a discrete boundary condition prescribed along the boundaries Γ_1 and Γ_2 that surround the total domain of the concave membrane, $\mathbf{U}_a$ and $\tilde{\mathbf{U}}_a$ represent unknown displacement vectors described at the nodes on Γ_a or $\tilde{\Gamma}_a$ corresponding to the common boundary of D_{I} and D_{II}, respectively; the participation vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ indicate the participation strengths of *NDIF*'s defined at the nodes on Γ_1 and Γ_2 respectively; $\mathbf{A}_a$ and $\tilde{\mathbf{A}}_a$ represent the participation strengths of *NDIF*'s defined at the nodes on Γ_a and $\tilde{\Gamma}_a$ respectively.

For the fixed boundary condition, i.e., $\mathbf{U}_1 = \mathbf{U}_2 = 0$, from equations (11, 13), the participation vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ are, respectively, related to $\mathbf{A}_a$ and $\tilde{\mathbf{A}}_a$ by

$$\mathbf{A}_1 = -\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a}\mathbf{A}_a, \qquad \mathbf{A}_2 = -\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2a}\tilde{\mathbf{A}}_a, \tag{15, 16}$$

which are, respectively, substituted into equations (12, 14). Then, the compatibility conditions on the common boundary Γ_a are utilized, i.e.,

$$\mathbf{U}_a = \tilde{\mathbf{U}}_a, \qquad \partial\mathbf{U}_a/\partial n = \partial\tilde{\mathbf{U}}_a/\partial n, \tag{17, 18}$$

where n denotes the normal direction from the common boundary. The substitution of equations (12, 14) into equations (17, 18) leads to

$$\mathbf{SM}(\Lambda)\mathbf{A} = \mathbf{0}, \tag{19}$$

where the system matrix $\mathbf{SM}(\Lambda)$ and the participation vector $\mathbf{A}$ are given by

$$\mathbf{SM} = \begin{bmatrix} \mathbf{SM}_{aa} - \mathbf{SM}_{a1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a} & \mathbf{SM}_{a2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2a} - \mathbf{SM}_{aa} \\ \mathbf{SM}'_{aa} - \mathbf{SM}'_{a1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a} & \mathbf{SM}'_{a2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2a} - \mathbf{SM}'_{aa} \end{bmatrix}, \quad \mathbf{A} = \begin{Bmatrix} \mathbf{A}_a \\ \tilde{\mathbf{A}}_a \end{Bmatrix} \tag{20, 21}$$

and $\mathbf{SM}'_{()}$ represents $\partial\mathbf{SM}_{()}/\partial n$. Finally, eigenvalues can be found by setting the determinant of the system matrix equal to zero, i.e., $\det[\mathbf{SM}(\Lambda)] = 0$.

3.2. THREE-DOMAIN METHOD: MEMBRANE WITH A HOLE

In order to solve a fixed membrane with a hole, the membrane is subdivided into several domains as shown in Figure 3, in which the three sub-domains D_{I}, D_{II} and D_{III} are surrounded by boundaries $\Gamma_1 + \tilde{\Gamma}_a + \Gamma_b$, $\Gamma_2 + \tilde{\Gamma}_b + \Gamma_c$ and $\Gamma_3 + \tilde{\Gamma}_c + \Gamma_a$ respectively. If the single-domain method is applied to each of the domains, three system matrix equations are obtained as

$$\mathbf{SM}_{\mathrm{I}}\mathbf{A}_{\mathrm{I}} = \mathbf{U}_{\mathrm{I}}, \qquad \mathbf{SM}_{\mathrm{II}}\mathbf{A}_{\mathrm{II}} = \mathbf{U}_{\mathrm{II}}, \qquad \mathbf{SM}_{\mathrm{III}}\mathbf{A}_{\mathrm{III}} = \mathbf{U}_{\mathrm{III}}, \tag{22–24}$$

which may be, respectively, rewritten as

$$\begin{bmatrix} \mathbf{SM}_{11} & \mathbf{SM}_{1a} & \mathbf{SM}_{1b} \\ \mathbf{SM}_{a1} & \mathbf{SM}_{aa} & \mathbf{SM}_{ab} \\ \mathbf{SM}_{b1} & \mathbf{SM}_{ba} & \mathbf{SM}_{bb} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_1 \\ \tilde{\mathbf{A}}_a \\ \mathbf{A}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_1 \\ \tilde{\mathbf{U}}_a \\ \tilde{\mathbf{U}}_b \end{Bmatrix}, \tag{25}$$

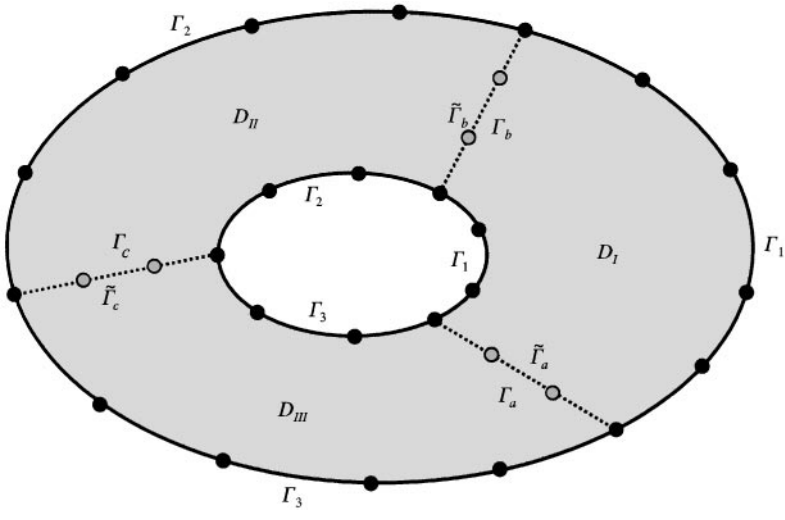

Figure 3. Multi-connected membrane subdivided into three domains D_{I}, D_{II} and D_{III}, surrounded, respectively, by the fixed and common boundaries, $\Gamma_1 + \tilde{\Gamma}_a + \Gamma_b$, $\Gamma_2 + \tilde{\Gamma}_b + \Gamma_c$ and $\Gamma_3 + \tilde{\Gamma}_c + \Gamma_a$.

$$\begin{bmatrix} \mathbf{SM}_{22} & \mathbf{SM}_{2b} & \mathbf{SM}_{2c} \\ \mathbf{SM}_{b2} & \mathbf{SM}_{bb} & \mathbf{SM}_{bc} \\ \mathbf{SM}_{c2} & \mathbf{SM}_{cb} & \mathbf{SM}_{cc} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_2 \\ \tilde{\mathbf{A}}_b \\ \mathbf{A}_c \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_2 \\ \tilde{\mathbf{U}}_b \\ \tilde{\mathbf{U}}_c \end{Bmatrix}, \tag{26}$$

$$\begin{bmatrix} \mathbf{SM}_{33} & \mathbf{SM}_{3c} & \mathbf{SM}_{3a} \\ \mathbf{SM}_{c3} & \mathbf{SM}_{cc} & \mathbf{SM}_{ca} \\ \mathbf{SM}_{a3} & \mathbf{SM}_{ac} & \mathbf{SM}_{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_3 \\ \tilde{\mathbf{A}}_c \\ \mathbf{A}_a \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_3 \\ \tilde{\mathbf{U}}_c \\ \tilde{\mathbf{U}}_a \end{Bmatrix}, \tag{27}$$

where $\mathbf{U}_1 = \mathbf{U}_2 = \mathbf{U}_3 = \mathbf{0}$ in the case of membranes with fixed edges.

From the first equation of the three equations that can be obtained from equation (25), the vector $\mathbf{A}_1$ may be expressed in terms of $\tilde{\mathbf{A}}_a$ and $\mathbf{A}_b$:

$$\mathbf{A}_1 = -\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a}\tilde{\mathbf{A}}_a - \mathbf{SM}_{11}^{-1}\mathbf{SM}_{1b}\mathbf{A}_b. \tag{28}$$

Similarly, from equations (26, 27),

$$\mathbf{A}_2 = -\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2b}\tilde{\mathbf{A}}_b - \mathbf{SM}_{22}^{-1}\mathbf{SM}_{2c}\mathbf{A}_c, \tag{29}$$

$$\mathbf{A}_3 = -\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3c}\tilde{\mathbf{A}}_c - \mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a}\mathbf{A}_a \tag{30}$$

Equations (28–30) are substituted into the remaining six equations that are not yet used in equations (25–27). Then, from the six equations, the six displacement vectors $\mathbf{U}_a$, $\tilde{\mathbf{U}}_a$, $\mathbf{U}_b$, $\tilde{\mathbf{U}}_b$, $\mathbf{U}_c$ and $\tilde{\mathbf{U}}_c$ may be expressed in terms of the six participation vectors $\mathbf{A}_a$, $\tilde{\mathbf{A}}_a$, $\mathbf{A}_b$, $\tilde{\mathbf{A}}_b$, $\mathbf{A}_c$ and $\tilde{\mathbf{A}}_c$ except $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$. Furthermore, the six slope vectors $\mathbf{U}'_a$, $\tilde{\mathbf{U}}'_a$, $\mathbf{U}'_b$, $\tilde{\mathbf{U}}'_b$, $\mathbf{U}'_c$ and $\tilde{\mathbf{U}}'_c$ may be obtained by differentiating the six displacement vectors with respect to the normal directions of the common boundaries.

Next, the compatibility conditions on the common boundaries

$$\mathbf{U}_a = \tilde{\mathbf{U}}_a, \qquad \mathbf{U}'_a = \tilde{\mathbf{U}}'_a, \tag{31, 32}$$

$$\mathbf{U}_b = \tilde{\mathbf{U}}_b, \qquad \mathbf{U}'_b = \tilde{\mathbf{U}}'_b, \tag{33, 34}$$

$$\mathbf{U}_c = \tilde{\mathbf{U}}_c, \qquad \mathbf{U}'_c = \tilde{\mathbf{U}}'_c, \tag{35, 36}$$

are applied to the six displacement vectors and the six slope vectors. Then, the above equations (31–36) are written in matrix form:

$$\mathbf{SM}(\Lambda)\mathbf{A} = \mathbf{0}, \tag{37}$$

where the system matrix $\mathbf{SM}(\Lambda)$ and the participation vectors $\mathbf{A}$ are given in Appendix A. Note that the order of the system matrix is determined as $2(N_a + N_b + N_c)$ where N_a, N_b and N_c denote the numbers of the boundary points distributed along the corresponding common boundaries. In the same manner as in the two-domain method, the eigenvalues can be found from $\det[\mathbf{SM}(\Lambda)] = 0$.

4. CASE STUDIES

In this section, several case studies are presented to verify the validity of the proposed method. For each case, the eigenvalues obtained are compared with those computed by exact analysis or FEM. The case studies show that the method is very effective when it is used for concavely shaped membranes with high concavity or multi-connected membranes with a hole.

4.1. RECTANGULAR MEMBRANE

First, the present method is applied to a rectangular membrane whose dimensions are $1{\cdot}2\text{ m} \times 0{\cdot}9\text{ m}$ (see Figure 4). For the discretization of the membrane, 27 nodes are distributed along the fixed boundary and the common boundary, for which three interior

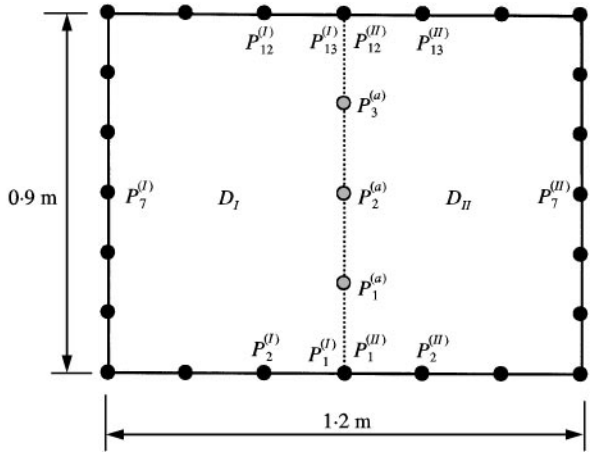

Figure 4. Rectangular membrane discretized with 27 nodes (24 boundary nodes and three interior nodes): only the boundary nodes are used for the single-domain method.

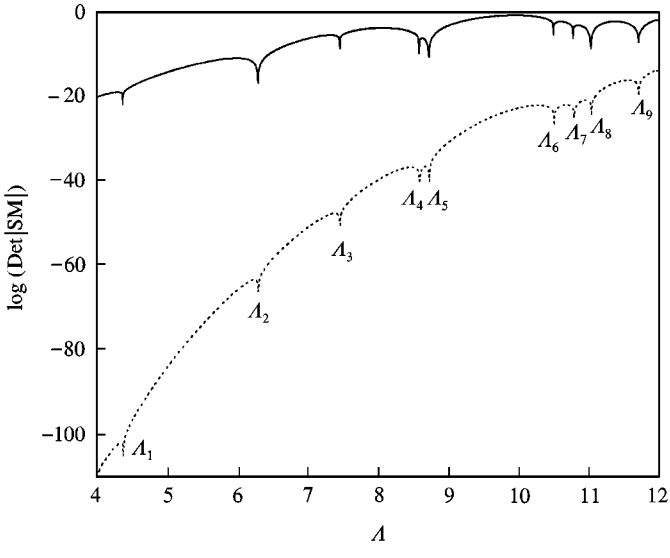

Figure 5. Determinant of the system matrix versus wavenumber for the rectangular membrane when the single- and two-domain methods are used. ——, two domains; - - - - -, single domain.

TABLE 1

Comparison of eigenvalues of the rectangular membrane obtained by the single-domain method, the two-domain method and FEM

Eigenvalues	Single domain	Two domains	Exact solution	FEM		
	24 nodes	27 nodes		1089 nodes	289 nodes	49 nodes
Λ_1	4·3633	4·3633	4·3633	4·3651	4·3703	4·4133
Λ_2	6·2929	6·2929	6·2929	6·3006	6·3240	6·5166
Λ_3	7·4560	7·4560	7·4560	7·4669	7·4996	7·7682
Λ_4	8·5948	8·5945	8·5947	8·6213	8·7013	9·1287
Λ_5	8·7266	8·7266	8·7266	8·7407	8·7828	9·3523
Λ_6	10·5083	10·5135	10·5083	10·5370	10·6234	11·3284
Λ_7	10·7943	10·7871	10·7943	10·8313	10·9428	11·8467
Λ_8	11·0389	11·0385	11·0384	11·1029	11·2974	12·7802

nodes are used. This membrane is solved by means of the single- and two-domain methods. In the case of the two-domain method, the system matrix $\mathbf{SM}_{\mathrm{I}}$ for D_{I} is made by using the 16 nodes located at the positions $P_1^{(\mathrm{I})} \sim P_{13}^{(\mathrm{I})}$ and $P_1^{(a)} \sim P_3^{(a)}$. The other system matrix $\mathbf{SM}_{\mathrm{II}}$ is made in the same manner as for $\mathbf{SM}_{\mathrm{I}}$, i.e., the 16 nodes located at the positions $P_1^{(\mathrm{II})} \sim P_{13}^{(\mathrm{II})}$ and $P_1^{(a)} \sim P_3^{(a)}$ are used. Note that the two points $P_1^{(\mathrm{I})}$ and $P_1^{(\mathrm{I})}$ or $P_{13}^{(\mathrm{I})}$ and $P_{13}^{(\mathrm{I})}$ are located at the same position.

The eigenvalues obtained by the two-domain method are represented by the values of the wavenumber Λ corresponding to the troughs in the logarithm curves shown in Figure 5. When the single-domain method is used, the eigenvalues are given by the previous study [1]. Those eigenvalues are summarized in Table 1 where the exact eigenvalues and the

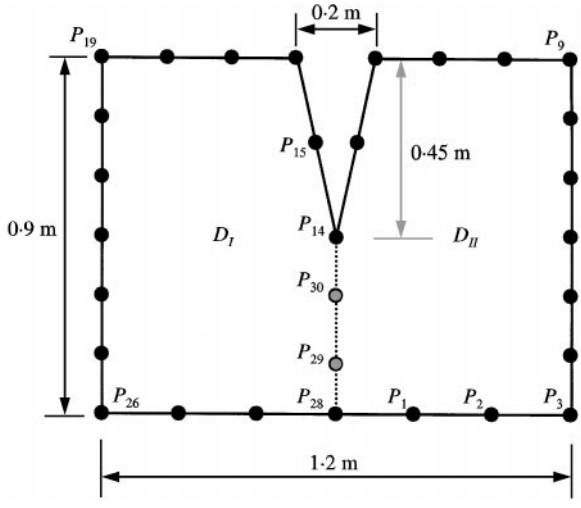

Figure 6. Concavely shaped membrane discretized with 30 nodes (28 boundary nodes and two interior nodes).

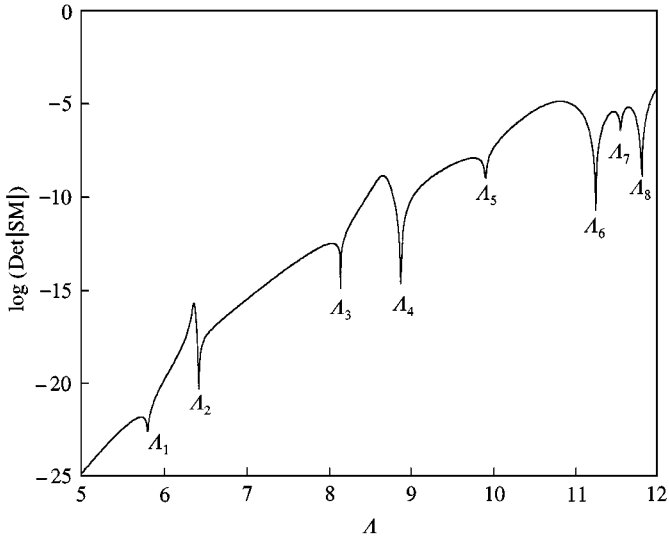

Figure 7. Determinant of the system matrix versus wavenumber for the concavely shaped membrane when the two-domain method is used.

eigenvalue computed by FEM (ANSYS) are also presented. It may be said that the two-domain method as well as the single-domain method gives very accurate eigenvalues. The first four eigenvalues obtained by the proposed method converge to the exact solution, and the other eigenvalues also converge nearly to the exact eigenvalues. It may be said that, if the eigenvalues obtained are compared with the eigenvalues obtained by FEM, the method guarantees rapid convergence in spite of the small number of boundary nodes. Note

TABLE 2

Comparison of eigenvalues of the concavely shaped membrane obtained by the two-domain method and FEM

Eigenvalues	Present 30 nodes	FEM		
		1701 nodes	976 nodes	451 nodes
Λ_1	5·79	5·71	5·72	5·74
Λ_2	6·42	6·42	6·43	6·44
Λ_3	8·15	8·17	8·18	8·21
Λ_4	8·88	8·89	8·90	8·92
Λ_5	9·92	9·87	9·89	9·94
Λ_6	11·25	11·31	11·34	11·43
Λ_7	11·55	11·46	11·48	11·56
Λ_8	11·81	11·84	11·86	11·92

that the 1089 nodes used in the FEM are sufficient to offer the first eight eigenvalues converged.

4.2. CONCAVE MEMBRANE WITH HIGH CONCAVITY

In order to obtain the eigenvalues of a concavely shaped membrane with high concavity, the two-domain method is applied to a rectangular membrane with a partially concave region, as shown in Figure 6. The common boundary illustrated by the dotted line is discretized with two interior nodes and the fixed boundary is discretized with 28 nodes respectively. Consequently, 30 nodes are used for the discretization of the membrane. Figure 7 shows the determinant curve where the values of the wavenumber corresponding to troughs represent the eigenvalues, summarized in Table 2. It may be said that the first eight eigenvalues obtained by the two-domain method are accurate compared with those obtained by FEM.

On the other hand, the single-domain method is also applied to the same membrane. For this purpose, the boundary of the membrane is discretized with 20, 24 and 28 nodes (see Figure 8). The determinant curves for the three discretized models are shown in Figure 9 where no dominant troughs are observed. It may be said from this fact that the single-domain method is not effective for concave membranes with high concavity.

4.3. L-SHAPED MEMBRANE

Figures 10(a)–(c) shows the discretized models for an L-shaped membrane. For two models (a) and (b), the total domain is divided into two domains of which the common boundary is discretized with three nodes. For model (c), the membrane is divided into three domains, and each of the two common boundaries is discretized with two nodes. 19, 27 and 28 nodes including the numbers of nodes on the common boundaries are used for the discretization of models (a), (b) and (c) respectively. Figure 11 shows the determinant curves from which the first eight eigenvalues are found. Those eigenvalues are summarized in

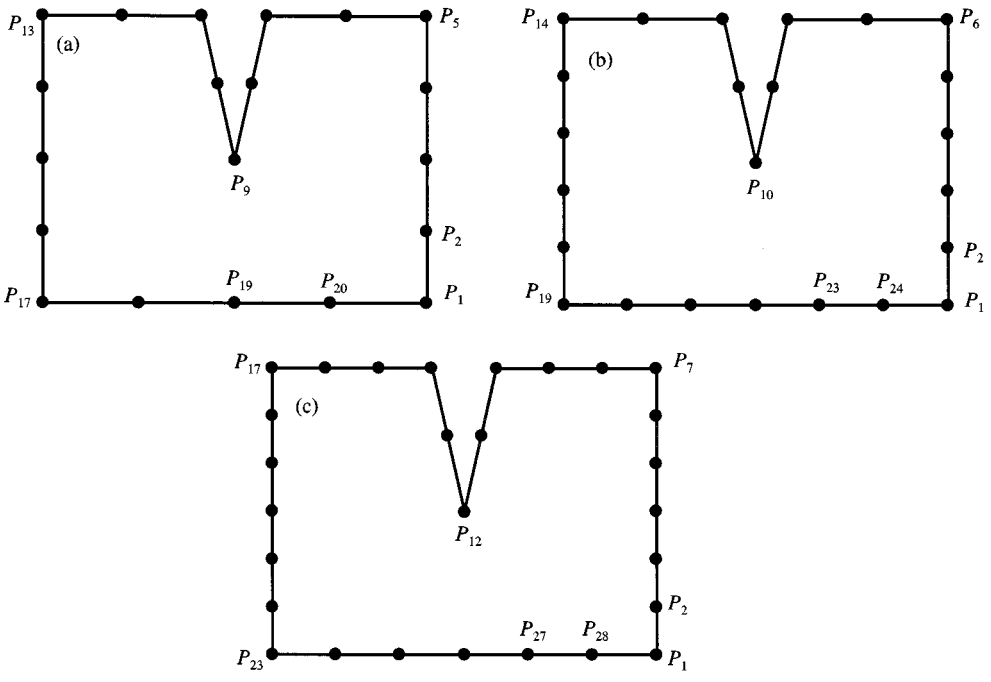

Figure 8. Discretized models of the concavely shaped membrane for the single-domain method when models (a)–(c) are discretized by 20, 24 and 28 boundary nodes respectively.

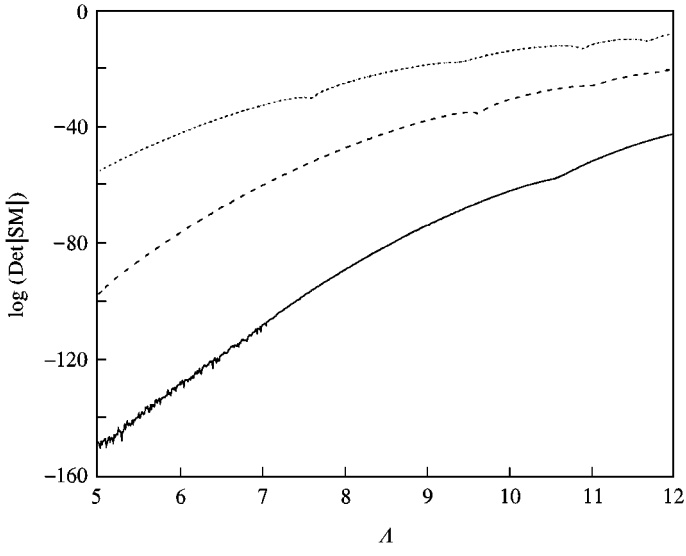

Figure 9. Determinant of the system matrix versus wavenumber for the concavely shaped membrane when the single-domain method is used. (Note that no dominant trough is observed). ·····, 20 nodes; – – – –, 24 nodes; ——, 28 nodes.

Table 3, which says that: (1) only the first six eigenvalues are found in the case of 19 nodes; (2) all the eigenvalues are obtained in the case of 27 nodes or 28 nodes; (3) the three-domain method yields the most accurate eigenvalues close to those computed by FEM.

TABLE 3

Comparison of eigenvalues of the L-shaped membrane obtained by the two-domain method, the three-domain method and FEM

Eigenvalues	Present method			FEM
	Two domains 19 nodes	Two domains 27 nodes	Three domains 28 nodes	1281 nodes
Λ_1	3·19	3·16	3·14	3·11
Λ_2	3·87	3·89	3·89	3·90
Λ_3	4·44	4·44	4·44	4·45
Λ_4	5·43	5·43	5·43	5·44
Λ_5	5·81	5·72	5·70	5·67
Λ_6	6·37	6·49	6·48	6·47
Λ_7	None	6·68	6·69	6·73
Λ_8	None	7·03	7·03	7·05

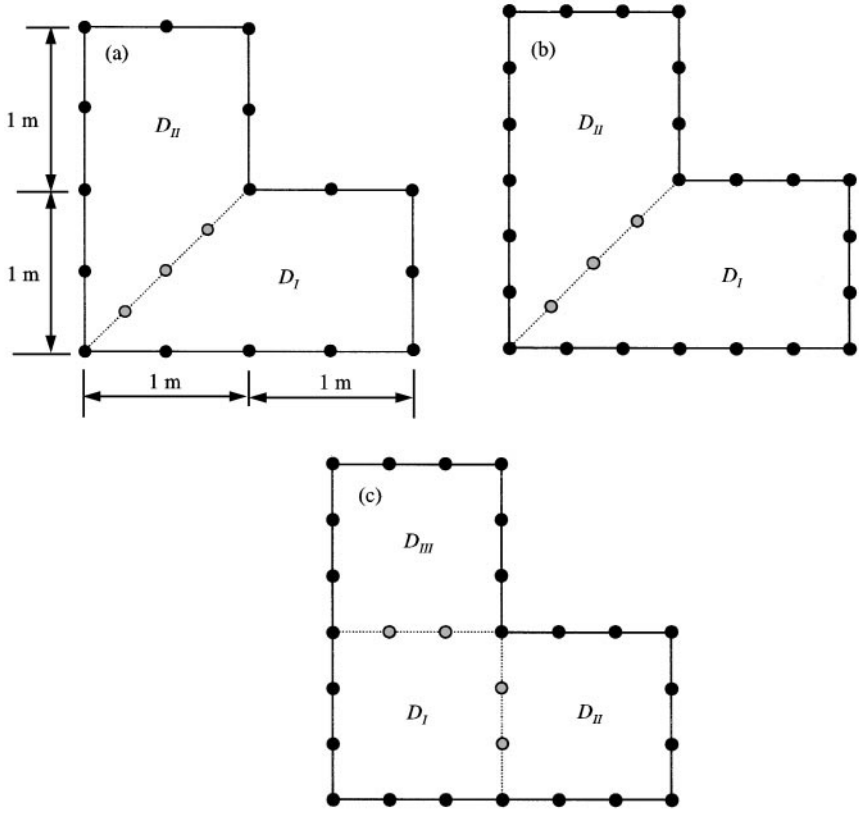

Figure 10. Discretized models of the L-shaped membrane for the two- and three-domain methods when models (a)–(c) are, respectively, discretized by 19, 27 and 28 nodes, including the numbers of interior nodes.

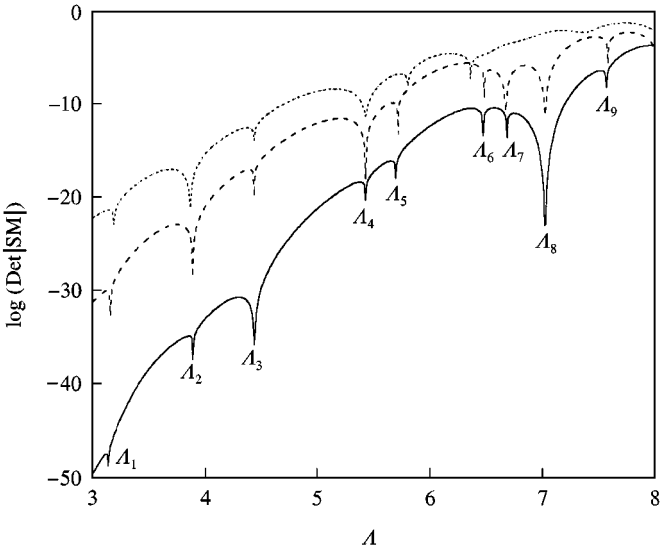

Figure 11. Determinant of the system matrix versus wavenumber for the L-shaped membrane when the two- and three-domain methods are used. ------, two domains (16 nodes); – – – –, two domains (24 nodes), ——, three domains (32 nodes).

TABLE 4

Comparison of eigenvalues of the multi-connected membrane obtained by the four-domain method and FEM

Eigenvalues	Present method			FEM
	28 nodes	40 nodes	52 nodes	880 nodes
Λ_1	3·79	3·71	3·71	3·67
Λ_2	3·97	4·00	4·00	3·99
Λ_3	4·44	4·51	4·50	4·53
Λ_4	5·04	5·23	5·27	5·19
Λ_5	5·80	5·87	5·90	5·89
Λ_6	6·46	6·46	6·48	6·51
Λ_7	6·63	7·02	7·03	7·07
Λ_8	6·67	7·13	7·12	7·24
Λ_9	None	7·23	7·23	7·39
Λ_{10}	None	7·77	7·77	7·73

4.4. SQUARE MEMBRANE WITH A HOLE

As shown in Figure 12, three kinds of discretized models are used to obtain the eigenvalues of a square membrane with a hole. Each of these models is divided into four domains of which the common boundaries are discretized with two nodes. Both the inner and outer boundaries of the membrane are assumed as fixed ones. The fixed boundaries of models (a), (b) and (c) are, respectively, discretized with 20, 32 and 42 nodes, 28, 40 and 52 nodes including eight interior nodes on the common boundaries are used for the discretization of models (a), (b) and (c) respectively. In Table 4, are summarized the

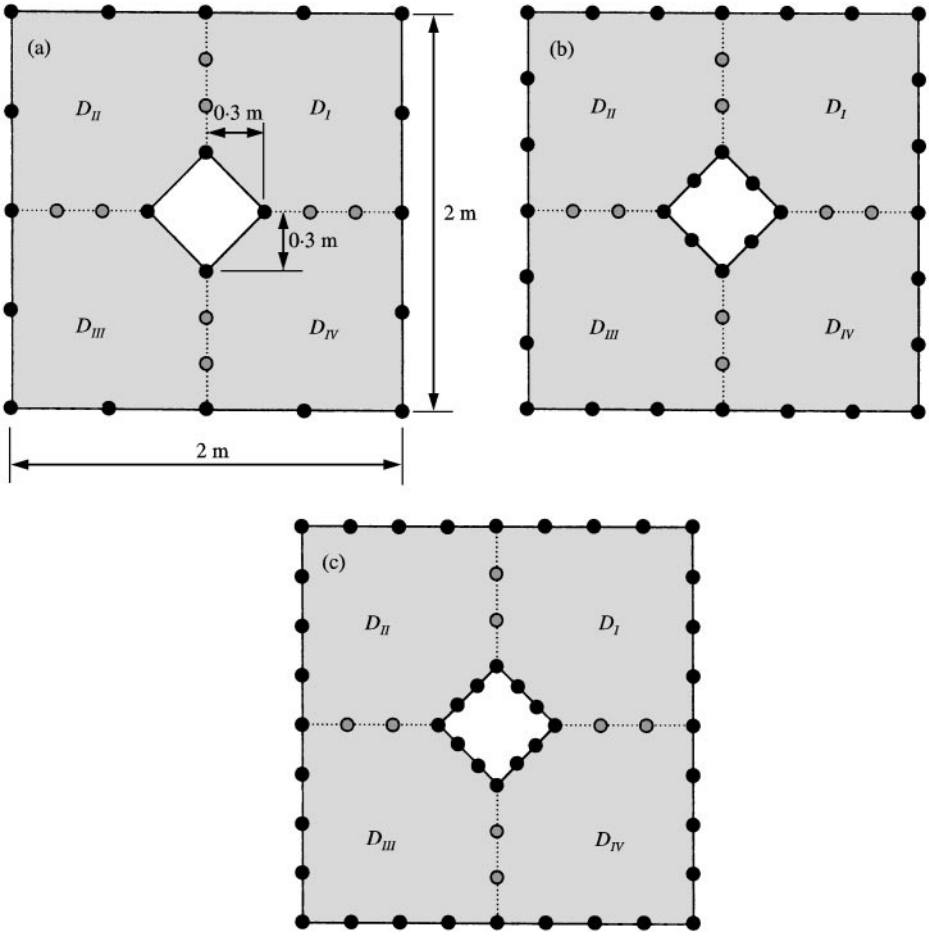

Figure 12. Discretized models of the multi-connected membrane for the four-domain method when models (a)–(c) are, respectively, discretized by 28, 40 and 52 nodes, including the number of interior nodes, 8.

eigenvalues obtained when the four-domain method is applied to the models. The concerned determinant curves are shown in Figure 13. For model (a), only the first six eigenvalues are close to the FEM results and it may be said, therefore, that more boundary nodes are required for higher eigenvalues. For models (a) and (b), it may be seen that all the eigenvalues are found to be close to the eigenvalues computed by FEM.

5. CONCLUSIONS

In this paper, a method, the multi-domain method, has been presented that can be used for free vibration analysis of arbitrarily shaped membranes with high concavity or holes. The method is more effective in calculating the frequency equation than the single-domain method, owing to the fact that the order of the system matrix depends on the number of interior nodes, not on the number of boundary nodes. Although the method is very simple in that no integration procedure to treat interpolation functions common to FEM and BEM is required, it gives accurate eigenvalues close to the eigenvalues obtained by FEM (ANSYS).

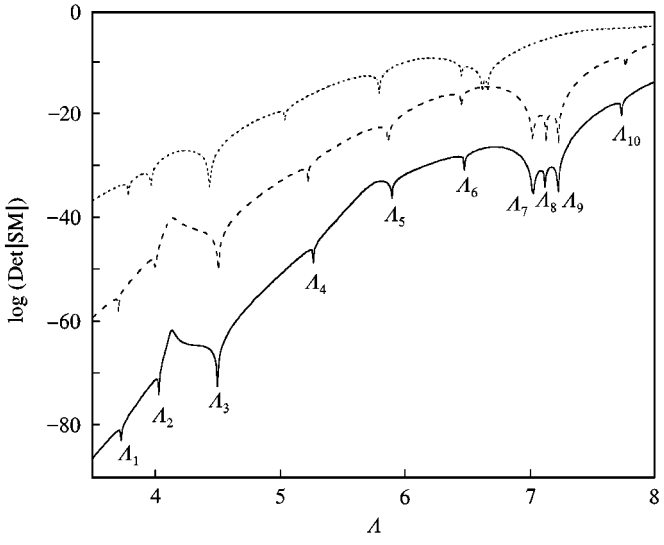

Figure 13. Determinant of the system matrix versus wavenumber for the multi-connected membrane when the four-domain method is used. -----, 20 nodes; – – – –, 32 nodes; ——, 44 nodes.

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APPENDIX A

Equation (37) is given by

$$\begin{bmatrix} \mathbf{SM}^{11} & \mathbf{SM}^{12} & \mathbf{0} & \mathbf{SM}^{14} & \mathbf{0} & \mathbf{SM}^{16} \\ \mathbf{0} & \mathbf{SM}^{22} & \mathbf{SM}^{23} & \mathbf{SM}^{24} & \mathbf{SM}^{25} & \mathbf{0} \\ \mathbf{SM}^{31} & \mathbf{0} & \mathbf{SM}^{33} & \mathbf{0} & \mathbf{SM}^{35} & \mathbf{SM}^{36} \\ \mathbf{SM}^{41} & \mathbf{SM}^{42} & \mathbf{0} & \mathbf{SM}^{44} & \mathbf{0} & \mathbf{SM}^{46} \\ \mathbf{0} & \mathbf{SM}^{52} & \mathbf{SM}^{53} & \mathbf{SM}^{54} & \mathbf{SM}^{55} & \mathbf{0} \\ \mathbf{SM}^{61} & \mathbf{0} & \mathbf{SM}^{63} & \mathbf{0} & \mathbf{SM}^{65} & \mathbf{SM}^{66} \end{bmatrix} \begin{Bmatrix} \mathbf{A}_a \\ \mathbf{A}_b \\ \mathbf{A}_c \\ \tilde{\mathbf{A}}_a \\ \tilde{\mathbf{A}}_b \\ \tilde{\mathbf{A}}_c \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix},$$

where

$$\mathbf{SM}^{11} = \mathbf{SM}_{aa} - \mathbf{SM}_{a3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a}, \qquad \mathbf{SM}^{12} = -\mathbf{SM}_{ab} + \mathbf{SM}_{a1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1b},$$

$$\mathbf{SM}^{14} = -\mathbf{SM}_{aa} + \mathbf{SM}_{a1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a}, \qquad \mathbf{SM}^{16} = \mathbf{SM}_{ac} - \mathbf{SM}_{a3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a},$$

$$\mathbf{SM}^{22} = \mathbf{SM}_{bb} - \mathbf{SM}_{b1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1b}, \qquad \mathbf{SM}^{23} = -\mathbf{SM}_{bc} + \mathbf{SM}_{b2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2c},$$

$$\mathbf{SM}^{24} = \mathbf{SM}_{ba} - \mathbf{SM}_{b1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a}, \qquad \mathbf{SM}^{25} = -\mathbf{SM}_{bb} + \mathbf{SM}_{b2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2b},$$

$$\mathbf{SM}^{31} = \mathbf{SM}_{ca} - \mathbf{SM}_{c3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a}, \qquad \mathbf{SM}^{33} = -\mathbf{SM}_{cc} + \mathbf{SM}_{c2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2c},$$

$$\mathbf{SM}^{35} = -\mathbf{SM}_{cb} + \mathbf{SM}_{c2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2b}, \qquad \mathbf{SM}^{36} = \mathbf{SM}_{cc} - \mathbf{SM}_{c3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3c},$$

$$\mathbf{SM}^{41} = \mathbf{SM}'_{aa} - \mathbf{SM}'_{a3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a}, \qquad \mathbf{SM}^{42} = -\mathbf{SM}'_{ab} + \mathbf{SM}'_{a1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1b},$$

$$\mathbf{SM}^{44} = -\mathbf{SM}'_{aa} + \mathbf{SM}'_{a1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a}, \qquad \mathbf{SM}^{46} = \mathbf{SM}'_{ac} - \mathbf{SM}'_{a3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a},$$

$$\mathbf{SM}^{52} = \mathbf{SM}'_{bb} - \mathbf{SM}'_{b1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1b}, \qquad \mathbf{SM}^{53} = -\mathbf{SM}'_{bc} + \mathbf{SM}'_{b2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2c},$$

$$\mathbf{SM}^{54} = \mathbf{SM}'_{ba} - \mathbf{SM}'_{b1}\mathbf{SM}_{11}^{-1}\mathbf{SM}_{1a}, \qquad \mathbf{SM}^{55} = -\mathbf{SM}'_{bb} + \mathbf{SM}'_{b2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2b},$$

$$\mathbf{SM}^{61} = \mathbf{SM}'_{ca} - \mathbf{SM}'_{c3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3a}, \qquad \mathbf{SM}^{63} = -\mathbf{SM}'_{cc} + \mathbf{SM}'_{c2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2c},$$

$$\mathbf{SM}^{65} = -\mathbf{SM}'_{cb} + \mathbf{SM}'_{c2}\mathbf{SM}_{22}^{-1}\mathbf{SM}_{2b}, \qquad \mathbf{SM}^{66} = \mathbf{SM}'_{cc} - \mathbf{SM}'_{c3}\mathbf{SM}_{33}^{-1}\mathbf{SM}_{3c}.$$